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MODELS FOR MINIMIZING THE FAILURE COSTS ASSOCIATED
WITH STATISTICAL QUALITY ACCEPTANCE SAMPLING PLANS

A THESIS

Presented to

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MODELS FOR MINIMIZING THE FAILURE COSTS
ASSOCIATED WITH STATISTICAL QUALITY
ACCEPTANCE SAMPLING PLANS

Approved:

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Date approved by Chairman 8/25/67

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TABLE OF CONTENTS

	Page
ACKNOWLEDGMENTS.	ii
LIST OF FIGURES.	v
SUMMARY.	vi
Chapter	
I. INTRODUCTION.	1
II. LITERATURE SURVEY	4
Introduction	
A Breakdown of Quality Costs	
Prevention Costs	
Appraisal Costs	
Failure Costs	
Total Quality Control	
Past Data as a Basis for Reducing Costs	
Optimum Total Quality Control Cost	
Statistical Acceptance Plans Based Upon Economic Criteria	
The Single Sampling Plan	
The Acceptable Quality Level Concept	
The Dodge-Romig System	
The System Designed by Hald	
Similarity of Research to that Done by Hald	
III. GENERAL APPROACH.	16
IV. A MINIMUM-COST PROCEDURE FOR HANDLING REJECTED ITEMS.	21
Introduction	
Producer Inspection	
Scrap Rejected Lots	
Sell Rejected Lots at Reduced Price	
Screen Rejected Lots	
Scrap Defective Items	
Sell Defective Items at Reduced Prices	
Rework Defective Items	
Scrap, Rework, and Reduced Price Combination	
The Minimum-Cost Procedure	
V. ESTIMATING THE AVERAGE NUMBER OF DEFECTS AND THE FREQUENCY OF EACH TYPE OF DEFECT IN REJECTED LOTS	30

Chapter	Page
V. ESTIMATING THE AVERAGE NUMBER OF DEFECTS AND THE FREQUENCY OF EACH TYPE OF DEFECT IN REJECTED LOTS (Contd.)	
Introduction	
Estimating the Average Number of Defects in Rejected Lots for Single Sampling Plans	
Case I	
Case II	
Prior Distributions	
Two-Point Binomial	
Hypergeometric	
Binomial	
Estimating the Average Number of Defects in Rejected Lots by Sampling and Screening Simulated Lots	
Prior Distribution	
Single Sampling Plan	
Procedure	
Results	
Analysis of Results	
Estimating the Frequency of Each Type of Defect in Rejected Lots	
Methods of Tabulating Data	
Adjusting the a_i 's for a Known Analytical Value of a	
A revised cost expression	
VI. MINIMUM-COST MODELS INCORPORATING CONSUMER BEHAVIOR	49
Consumer Behavior	
Case 1	
Probability P(R) of Obtaining a Rejected Lot	
Expected (Average) Number of Defects in an	
Unacceptable Lot	
Minimizing the Total Failure Cost for Case 1	
The OC-curve and Parameters of the Single Sampling Plan	
The Minimum-Cost Procedure	
Case 2	
Case 3	
Case 4	
Case 5	
VII. CONCLUSIONS AND RECOMMENDATIONS	66
BIBLIOGRAPHY	68

LIST OF FIGURES

Figure	Page
2-1. Economics of Quality of Conformance	8
3-1. Decision-Action Diagram	17
5-1. Average Number of Defective Items per Rejected Lot as a Function of the Number of Lots Sampled (Case I)	41
5-2. Average Number of Defective Items per Rejected Lot as a Function of the Number of Rejected Lots Screened (Case I).	42
5-3. Average Number of Defective Items per Rejected Lot as a Function of the Number of Rejected Lots Screened (Case II)	43
5-4. Proposed Data Tabulation Sheet--Lot Tally Sheet	46
5-5. Proposed Data Tabulation Sheet--Summary Sheet	47
6-1. OC-Curve for the Single Sampling Plan with $n = 100$ and $c = 2$	58
6-2. OC-Curves for Different Sample Sizes.	60
6-3. OC-Curves for Different Acceptance Numbers.	60

SUMMARY

The basic field in which the research is conducted is quality control, in particular the economical aspects of statistical acceptance sampling. The study is concerned exclusively with failure costs--those costs incurred by a manufacturer due to material which does not meet company or buyer specifications. The purpose of the research is to develop models to minimize total failure cost for specific producer-consumer relationships without affecting other costs related to quality control.

The environmental basis upon which the study is carried out is illustrated by a decision-action diagram (Figure 3-1, page 17) which describes an arbitrary set of futures which a finished item might undergo. The diagram incorporates both possible producer and consumer behavior. It is assumed that produced material is initially in lots of a fixed size, and failure costs will be expressed on a per lot (or per lot size) basis.

The first step in the study is to develop a procedure for minimizing the cost of handling lots rejected by the producer's statistical acceptance plan. As shown by the decision-action diagram, the producer has three alternatives open to him regarding the disposition of rejected lots--scrap them, sell them at a reduced price, or screen them and take subsequent action according to the diagram. The losses in revenue from scrapping rejected lots or selling them at a reduced price are regarded as "costs" of having defective material, and assumed to be independent of

the amount of defective material in the lots. For the screening alternative the total failure cost per lot depends on the number and different types of defects in the lot. A simple minimum-cost procedure based on inspection of scrap values, reduced prices, and rework costs for the different types of defects is specified for the screening alternative. The minimum cost for the screening alternative is compared with the alternative costs of scrapping lots or selling them at a reduced price, and the lowest-cost alternative is selected.

An analytical method of determining the average number of defective items in rejected lots is developed, based on knowledge about the prior distribution of lot fraction defective. Monte Carlo simulation techniques are used to simulate lots of product, which are sampled according to a specified single sampling plan. The value for the average number of defects in rejected lots for the simulated case is compared to the value obtained analytically, as a cumulative function of the number of lots sampled and the number of rejected lots screened, respectively, the purpose being to test the closeness of sample data to the analytical value.

The minimum-cost procedures for handling rejected material are utilized in cost models which determine the total failure cost *per lot produced* for different producer-consumer situations illustrated by the decision-action diagram. Five cases are considered, and cost models are formulated for two of them. These two cases are:

1. The case where the consumer inspects incoming lots and returns rejected lots to the producer.

2. The case where the consumer inspects incoming lots, screens rejected lots, and returns the defective items obtained by screening to the producer.

Analytical expressions are developed to determine the total failure cost per lot produced (wherein the minimum-cost procedures for handling rejected material are extended to defective material returned by the consumer) for each of the two cases. Given knowledge about the parameters (n, c) of the consumer's inspection plan (assumed to be a single sampling plan), the failure cost per lot produced is plotted graphically as a function of the acceptance number c of the producer's inspection plan (also assumed to be a single sampling plan), and the value of c for the producer's plan is selected which will minimize the total failure cost. Since changing the acceptance number of the plan does not significantly change the cost of carrying out the inspection process, the failure cost will have been minimized without affecting the other costs of inspection.

A number of recommendations are made regarding possible subsequent research in the same area, including:

1. The formulation of a computer program to determine the values of the failure costs for the different acceptance numbers and select the minimum-cost inspection plan for the producer.

2. The extension of the models to include the effects of changing the sample size n in the producer's plan, thus bringing into consideration the costs of carrying out the inspection, which to some degree will be a function of the sample size.

CHAPTER I

INTRODUCTION

One of the major functions of quality control is acceptance sampling, and a considerable amount of literature has been devoted to the subject. Much of the early literature dealt with the design and execution of various types of statistical sampling plans to yield quality assurance. In recent years, increasing emphasis has been placed on the economic aspects of acceptance sampling procedures.

One approach toward classifying the costs of quality control has been that taken by Masser,¹ who divides the costs of quality into three major categories: prevention costs, appraisal costs, and failure costs. *Prevention costs* are defined as those costs incurred in the process of keeping defects from occurring in the first place, including such cost areas as quality control engineering and quality training. *Appraisal costs* include the expenses for maintaining company quality levels by means of formal evaluations of product quality; e.g., the costs of carrying out the inspection processes and maintaining inspection equipment. *Failure costs* are caused by defective materials and products that do not meet company or buyer specifications. Included in this category are such items as scrap, rework, and the cost of vendor relations on defective items. These three types of costs will be

¹W. J. Masser, "The Quality Manager and Quality Costs," *Industrial Quality Control*, vol. 14, 1957, pp. 5-8.

discussed more extensively in the Literature Survey.

The costs of acceptance sampling are those encompassed by the last two categories mentioned above, namely, appraisal and failure costs. Attempts have been made by various authors to design statistical inspection plans such that the sum of appraisal and failure costs will be minimized. Several of these minimum-cost procedures are presented in the Literature Survey. This research will be concerned exclusively with *failure* costs, and procedures for defining and minimizing the total failure cost in particular situations without affecting appraisal costs.

The failure costs incurred by a producer, disregarding the economic effects of loss of good will, will depend upon the amount of defective material which the producer has on hand, and the procedures which he adopts with respect to this material. The amount of unacceptable material will be influenced by his own standards toward the quality of material; i.e., his acceptance sampling procedures, and by consumer policy toward purchased material. The course of action which the producer takes with defective material may be fixed or open to selection in the case of defective material possessed by the consumer, and is entirely open to selection in the case of defective material obtained by the inspection process.

In the study, limitations will be imposed upon:

1. The nature of the product.
2. The nature of the producer's inspection process.
3. The range of alternative courses of action open to the producer with respect to defective material.

4. The range of alternatives open to the consumer regarding incoming and defective material.

Within the framework of these limitations, a decision-action diagram will be constructed which will describe the possible futures which may befall a lot of finished product. This diagram will be discussed in greater detail later in the report. The purpose of this study will be to minimize the *failure cost* (without affecting appraisal costs) of the producer, for a number of producer-consumer situations illustrated by the decision-action diagram, by:

1. Developing a decision process which will result in handling of defective material at a minimum cost to the producer.

2. Establishing procedures for minimizing the total amount of defective material which the producer has on hand, for each of the producer-consumer situations considered.

The approach adopted in this study is somewhat different from those utilized by previous researchers. It is hoped that the results of the research will provide a new and useful, although limited technique for minimizing failure cost.

CHAPTER II

LITERATURE SURVEY

Introduction

The costs of achieving and maintaining quality in industry have become very high in recent years. Evidence points to the fact that many businesses have quality-cost expenditures representing 7, 8, or 10 per cent, or even more, of their cost of sales.

There are two challenges that competitive conditions present to American management with regard to product quality:

1. Considerable improvement in the quality of many products and many quality practices.
2. Substantial reduction in overall costs of maintaining conformance to a specified level (or levels) of quality.

It is within this second general area of cost reduction that this study will take place.

A Breakdown of Quality Costs

There are numerous approaches to the concept of quality control cost. As discussed in the Introduction, one approach is that taken by Masser.¹

Masser separates quality control costs into prevention costs, appraisal costs, and failure costs. Prevention and appraisal costs

¹Masser, *op. cit.*, pp. 5-8.

will be considered briefly, and failure costs will be discussed more extensively, because of their particular relevance to this study.

Prevention costs are spent for the purpose of keeping defects from occurring in the first place. Included in this area are such costs as quality engineering, quality maintenance of patterns and tools, employee quality training, and writing quality assurance instructions.

Appraisal costs include the expenses for maintaining quality levels by means of formal evaluations of product quality. This involves such cost elements as inspection, test, quality audits, outside endorsements, and laboratory acceptance examinations.

Failure costs are caused by defective materials and products that do not meet company quality specifications. Masser considers failure costs to consist of the following elements:

1. *Scrap*--all losses incurred for scrap, excepting that due to the fault of the vendor.
2. *Rework*--all losses incurred for rework, excepting that due to the fault of the vendor.
3. *Scrap and rework--fault of vendor*--all losses to the business incurred by the use of vendor material that does not conform to specifications. This includes both rework and scrap, regardless of whether the material is accepted on a "use as is" basis or whether it is within the acceptable quality level.
4. *Material procurement*--an average cost to the business for outside vendor or allied plant inspector's rejection, multiplied by the number of required inspection reports issued for the period covered.

Also included is the cost of issuing formal complaints to vendors for any lots of questionable material accepted on a review basis.

5. *Factory contact engineering*--costs due to the time design or product engineering personnel spend on factory problems involving quality. Exclude engineering development work.

6. *Complaints*--all expenditures for the adjustment of complaints.

7. *Product services*--all product service charges directly attributable to correcting imperfections, not included in complaints.

The basic concept of failure costs--the costs of defective material--will be adopted in this study. However, the elements comprising total failure cost will be altered for the purposes of the study, while still conforming to the definition of failure costs.

Feigenbaum¹ states that the breakdown of total quality cost can be generalized as follows: prevention costs--5 per cent, appraisal costs--25 per cent, failure costs--75 per cent.

Total Quality Control

Feigenbaum suggests the "total quality control" approach to reducing quality control costs. Emphasis is placed on starting control with the design of the product and ending only when the product has been placed in the hands of a customer who is satisfied. It is his belief that a deliberate increase in prevention costs will result in even greater savings in appraisal and prevention costs. Tightening inspection procedures (and thereby increasing appraisal costs) does not

¹A. V. Feigenbaum, *Total Quality Control*, New York, McGraw-Hill Book Company, Inc., 1961, p. 84.

effectively reduce the number of defects, and the best procedure is to spend additional money on emphasizing prevention of defects before they occur.¹

Past Data as a Basis for Reducing Costs

Past data is often used as a basis for future actions aimed toward reducing quality control costs. Quality control reports, which may be prepared by the accounting department, illustrate the percentage breakdown of the various costs, using some type of volume base such as sales, direct labor, or manufacturing-cost output. Management will then select the larger percentage categories during this time period as primary areas for cost reduction during the ensuing time period.²

Optimum Total Quality Control Cost

Juran³ states that there is an optimum total quality control cost, which is depicted graphically in Figure 2-1. Increased quality conformance reduces the losses due to defectives. However, the cost of the controls needed for greater conformance rises geometrically as perfection (zero defects) is approached. The optimum is always short of perfection.

¹*Ibid.*, p. 16.

²*Ibid.*, p. 89.

³J. M. Juran (Editor), *Quality Control Handbook*, New York, McGraw-Hill Book Company, Inc., 1951, p. 1-10.

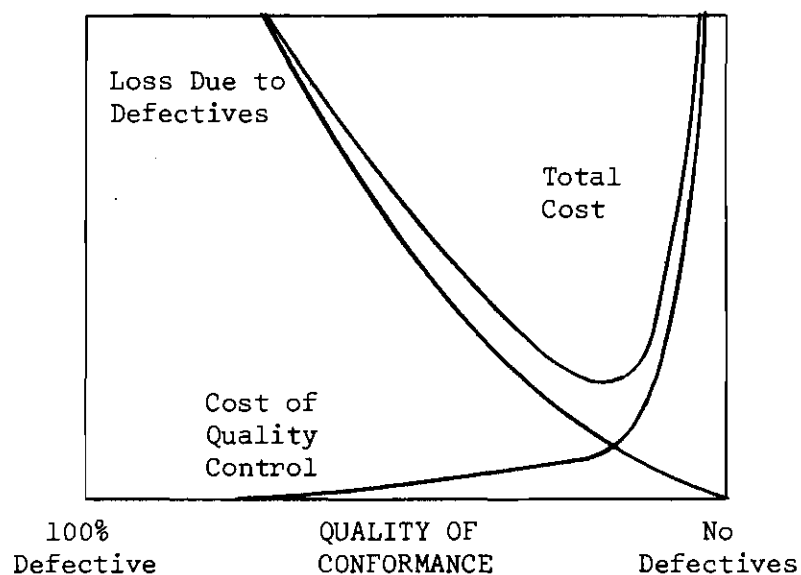


Figure 2-1. Economics of Quality of Conformance

Statistical Acceptance Sampling Plans

Based Upon Economic Criteria

Three factors which have been important in the design of statistical acceptance sampling plans are the operating characteristics and the producer's and consumer's risks. Dodge and Romig¹ first introduced the concepts of the lot tolerance fraction defective, the consumer's risk, the process average and the average amount of inspection, and illustrated means of expressing these quantities by the lot size, the sample size, and the acceptance number. Since then, the most popular and useful concept has been the operating characteristic.

The Single Sampling Plan

A single sampling plan is defined by three numbers: the lot size

¹H. F. Dodge and H. G. Romig, *Sampling Inspection Tables--Single and Double Sampling*. New York: John Wiley & Sons, Inc., 1962.

N , the sample size n , and the acceptance number c . The decision rule is to accept the lot if the number of defectives in the sample is equal to or less than the acceptance number, otherwise reject the lot. For a lot with fraction defective $p = X/N$, the probability of acceptance is¹

$$P_a(p) = \sum_{x=0}^c \frac{\binom{X}{x} \binom{N-X}{n-x}}{\binom{N}{n}} \quad (2-1)$$

Plotting the acceptance probability as a function of the fraction defective of the inspection lot results in the operating characteristic (OC) curve of the given sampling plan, which completely describes the power of the plan to discriminate between lots with high and low fractions defectives, respectively.

A common method of designing the inspection plan has been to choose two points on the OC-curve which lead to two equations for the determination of n and c . These two points are usually known as the producer's and consumer's risk points. The producer's risk point $(p_1, 1-\alpha)$ consists of a low fraction defective and a high probability $(1-\alpha)$ of acceptance; i.e., the producer wishes to reject lots of an acceptable quality level p_1 only α per cent of the time. The consumer's risk point (p_2, β) consists of a high fraction defective p_2 and a low probability β of acceptance. For example, n and c may be set as closely

¹A. Hald, "The Compound Hypergeometric Distribution and a System of Single Sampling Inspection Based on Prior Distributions and Costs," *Technometrics*, vol. 2, no. 3, 1960, p. 277.

as possible (satisfying integer requirements) to a plan with $\alpha = .05$ and $\beta = .10$.

Hald states that:

Among the practical considerations in determining the risk points the following may be mentioned: What quality has the consumer actually got previously, i.e. what is the (prior) distribution of previously submitted lots? What is normal market quality at the price the consumer is willing to pay, i.e. what are the (prior) distributions for the suppliers of the market? What fraction defective can the consumer tolerate without giving him any essential trouble with the intended use of product and what fraction defective will be intolerable, i.e. how does the trouble (damage, loss) depend on number of defective items accepted?

Besides these points other considerations, as for example the need for the good in question, comes in. The conclusion is that even if we have a perfect solution to the purely statistical problem of determining a sampling plan corresponding to any given set of two points on an OC-curve the resulting plan must to a large extent be considered arbitrary because we have no rational way of choosing the producer's and consumer's risks and risk points.¹

Thus there are no clear-cut considerations for economical choice of the two risk points from which the sampling plan may be devised.

The Acceptable Quality Level Concept

An important concept in some sampling systems is the Acceptable Quality Level (AQL), which may be defined in the following way: "The maximum percent defective which can be considered satisfactory as a process average."² Hald states that it can be shown that "under some simple assumptions regarding costs *it does not pay to inspect if the lots submitted for inspection are produced from a process which is in statistical control with a process average p less than or equal to the*

¹*Ibid.*, p. 278.

²Industrial Quality Control, 14, no. 5, 1957, p. 6.

AQL, i.e. a binomial prior distribution."¹ It is assumed that the loss caused by accepting a defective on the average is 1; i.e., the loss from accepting a defective is used as a unit for the other cost elements, and that the inspection cost per item inspected is k . The cost of 100 per cent inspection then becomes Nk and the loss from (cost of) no inspection is Np . Assuming $p < k$, the average total cost of sampling inspection is shown to be (assuming rejected lots to be fully inspected)

$$nk + (N-n)(pP_a + k(1-P_a)) \quad (2-2)$$

which is always larger than Np for $p < k$.² Thus the AQL concept implies some prior cost considerations.

The Dodge-Romig System

Numerous methods have been developed for specific situations whereby one of the two points is fixed and the inspection plan is designed so as to minimize costs while satisfying the restriction. Dodge and Romig utilized such an approach.²

In the Dodge-Romig paper, the discussion is limited to non-destructive inspection and it is further specified that all rejected lots will be completely inspected and all defective items replaced by good items. The authors consider a situation with a prior distribution composed of a normal part, which is binomial with process average p , and

¹Hald, *op. cit.*, p. 279.

²Dodge and Romig, *op. cit.*

a part of considerably poorer quality, without specifying the form of the poor part or the relative weights of the two parts. A consumer's risk point (p_t, β) is specified, such that p_t is the "lot tolerance fraction defective"--that fraction defective which represents a completely unacceptable lot--and β , set equal to .10, satisfies customer desire to be protected from bad lots.¹

Basically, the Dodge-Romig system seeks to minimize the average total inspection cost for normal production and at the same time give the required consumer protection. Using the cost of inspecting an item as unit, the cost function becomes equal to the average amount of inspection, which may be written as

$$I = n + (N-n)(1-P_a) \quad (2-3)$$

The optimum values of (n, c) are then determined as the values minimizing $I(p)$ under the restriction that $P_a(p_t) = .10$.²

In his remarks on the Dodge-Romig system of minimizing inspection costs, Hald states:

Let us now consider the Dodge-Romig system within a somewhat wider framework of assumptions. Presumably they have chosen their rather vaguely specified prior distribution partly because information on prior distributions is scarce and partly to keep the number of parameters down. Even if the prior distribution formally is represented in the system only by the one parameter p it is obvious that the prior distribution also influences the choice of p_t . It must be assumed that the prior distribution extends on both sides of p_t . Further the choice of p_t must also depend on cost considerations which are not taken explicitly into

¹Hald, *op. cit.*, p. 282.

²*Ibid.*, p. 282.

account in the system. Instead of taking all cost elements into consideration Dodge and Romig limit themselves to *inspection* costs only presumably because these are the most easily accessible. Even if the concept of a loss resulting from the acceptance of defective items is not explicitly introduced it must be tacitly assumed that for lots of tolerance quality it would be cheaper to sort the whole lot than to accept the lot without inspection whereas for lots of process average quality the opposite is true . . . The cost function considered by Dodge and Romig is a monotone function of n and c and therefore (without any restrictions on n and c) leads to the conclusion that minimum inspection costs for product of normal quality are obtained by acceptance without inspection. To reach an optimum *sampling* plan which minimizes the average total inspection cost for normal production Dodge and Romig therefore has to introduce an (arbitrary) relation between n and c which is achieved by means of the above-mentioned condition that the OC-curve shall pass through a given point.¹

The System Designed by Hald

Hald considers the design of a single sampling plan based on the prior distribution of lot fraction defective and the economic consequences of rejection and acceptance. Initially, two basic assumptions are made:

1. Defective items found by inspection (either sampling or sorting) are replaced by effectives.
2. The average loss (due to servicing, rework, etc.) caused by an accepted defective item is used as the economic unit for the evaluation of other cost elements; i.e., it is given a value of 1 and all the other costs are expressed in terms of it. Thus the necessity of economically evaluating this cost is eliminated.

In addition to the case of sampling inspection, three limiting cases are also considered--acceptance without inspection, rejection

¹*Ibid.*, p. 282.

without inspection (all lots screened), and the situation where the quality of each submitted lot is known. The last case corresponds to the minimum cost.

For the sampling case, Hald formulates an expression for the average total cost of inspection. This expression is rather complex, and will not be included here, but basically it is related to the nature of the prior distribution, rather than the process average and producer's and consumer's risk points. Hald devotes the remainder of his article to discussing prior distributions and means of evaluating those terms in the average total cost expression which are dependent upon the prior distribution. The objective is to select an inspection plan with (n, c) such that the average total cost of inspection is minimized.¹

Similarity of Research to That Done by Hald

The concepts introduced by Hald regarding prior distributions have been quite useful to this writer in the development of minimum-cost procedures. Some of them will be discussed later in the report.

Hald and other authors generally specify some arbitrary course of action with regard to rejected lots, such as screening or sorting the lots and replacing defectives with good items. One objective of this research, as mentioned earlier, is to specify a means of selecting this course of action such that the resultant cost is minimized. Also, in this research the cost of accepting a bad item will be analogous to

¹*Ibid.*, pp. 282-340.

the cost of handling a defective item obtained by inspection.

The approach adopted in this research is similar to that taken by Hald in that it emphasizes the importance of the prior distribution, but different in its conception of the costs of handling defective items. Also, the research is focused primarily upon failure costs, rather than the total cost of inspection. Basically, the intent of this research is to develop a new method of minimizing failure costs for certain situations, utilizing earlier concepts developed by Hald.

CHAPTER III

GENERAL APPROACH

It is necessary first to describe the environmental conditions upon which this study is based. The primary basis for the study is the decision-action diagram (Figure 3-1) on page 17.

In the diagram, the boxes represent courses of action, and the circles represent decision points. The diagram is intended to illustrate the possible courses that a finished item may take after it has left the production line. The diagram does not necessarily include all the possible actions to which an item might be subjected, but it is believed that the majority of real life possibilities are encompassed. For the purposes of the study, only those courses of action which are illustrated by the decision-action diagram will be considered.

A number of conditions are set down regarding the nature of the product. It is assumed that the product:

1. Is of a single type.
2. Is mass-produced and reaches the final inspection stage in lots of a predetermined size.
3. Is such that inspection will not be destructive.
4. Requires no producer servicing in the field; i.e., the consumer must do any rework himself, but may return defective material.

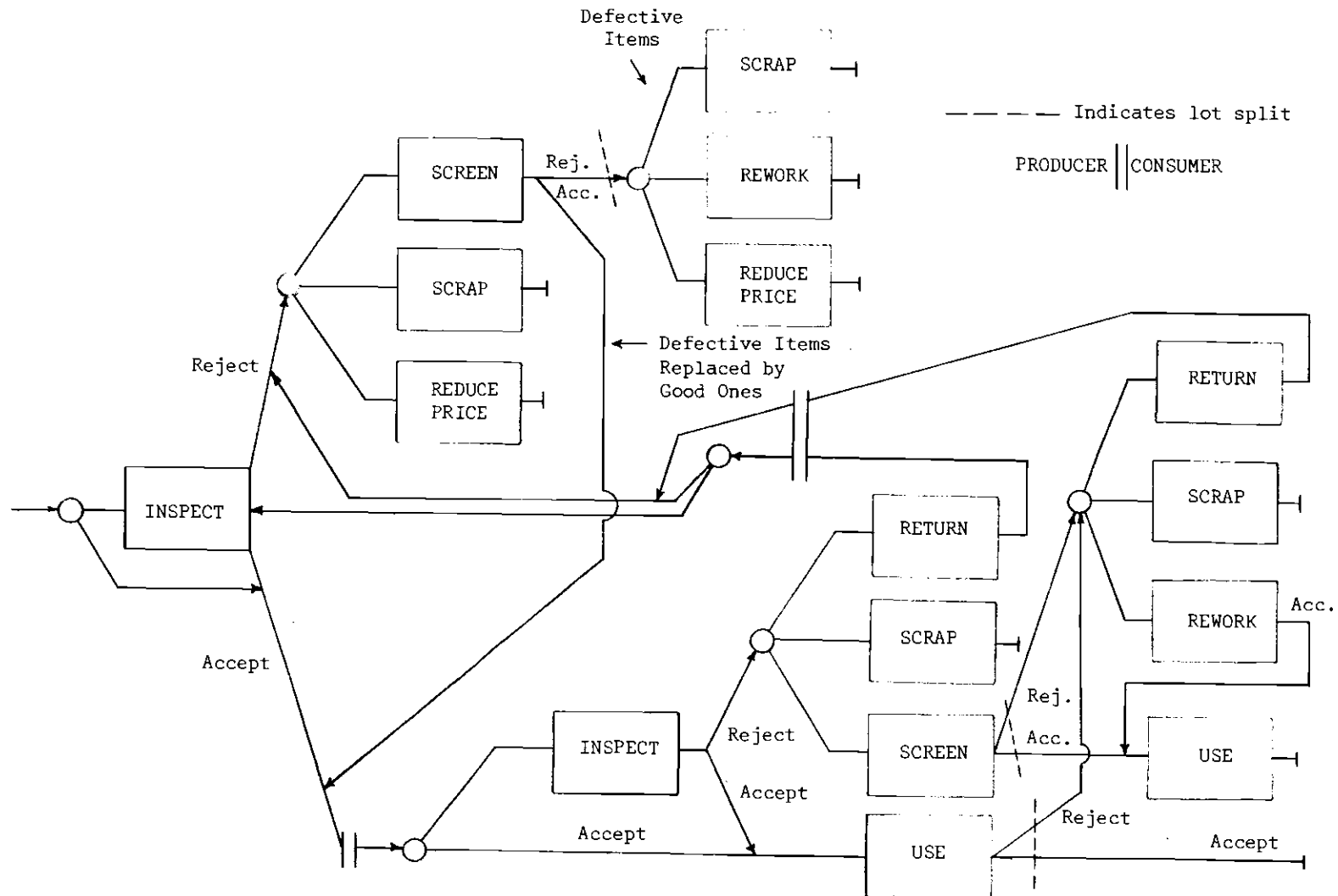


Figure 3-1. Decision-Action Diagram

The study is concerned with the final product, from the point of completion of production operations. Any subinspections that may have occurred prior to the completion of production are not considered.

The producer first has the choice of subjecting lots of finished items to some form of final inspection, or accepting them as is. If he decides to inspect the lots according to some acceptance sampling plan, some of the lots will be rejected, according to the acceptance criteria of the plan. The producer must now make another decision as to what to do with the rejected lots. As shown on the diagram, he may sell the lots at a reduced price, screen them for defects, or scrap them. If he elects to screen the lots, he is again faced with several alternatives as to what to do with the defective items. It may thus be seen that an item which is part of a lot may follow a number of different paths through the diagram.

The consumer faces similar decisions regarding incoming lots of finished items. He may inspect them himself, according to some plan which may or may not be similar to that of the producer, or accept the lots and proceed with the use of the items. The subsequent decision processes of the consumer are similar to those of the producer. However, the consumer has the option of returning unacceptable lots or items to the producer.

If the consumer is satisfied with the overall quality level of the incoming product; i.e., he will continue to purchase from the producer, then those lots which are accepted by the producer and sent to the consumer are of no further economic consequence to him, unless the lots (or items from the lots) are returned to him by the consumer.

In the event that unacceptable lots or items are returned to him, the decision processes are again available to him regarding the disposition of the returned material.

The first objective of this study is to develop a procedure for determining the least-cost method of handling rejected items, for those alternatives presented by the decision-action diagram. It should be emphasized that the study *assumes* rejected material; i.e., it is not concerned with any costs up to and including final inspection, but only those incurred through the handling of defective lots or items.

A simple model will be developed, which will provide a means of identifying and minimizing the costs of handling defective material. The model will utilize data which is already available to the producer or relatively easy to obtain. It is primarily a means of determining the cost of each alternative open to the producer regarding defective material, so that the minimum-cost alternative may be selected by inspection.

The rejected material which the producer has on hand at a given time may originate from two sources. His own acceptance procedure may result in rejected lots, and lots or items may also be returned to him by the consumer. For example, the consumer may decide to inspect incoming lots, and return all rejected lots to the producer.

It will be assumed that the producer utilizes a single sampling plan to inspect lots. With this assumption in mind, each possible case where defective material may be returned to the producer (as shown by the decision-action diagram) will be considered. In those cases considered to be most likely to exist in real life, an analytical method

will be developed for predicting the expected amount of unacceptable material which the producer will have on hand.

Given the methods described above, and the minimum cost model discussed earlier, procedures will be specified for minimizing the *total failure cost per lot produced*, in each case considered to be of importance.

CHAPTER IV

A MINIMUM-COST PROCEDURE FOR HANDLING REJECTED ITEMS

Introduction

This chapter is concerned with a minimum-cost decision procedure for the handling of rejected items. As stated before, defective items may either be detected by the producer's inspection process or be returned to him by the consumer. The first case--that of defective items detected by the producer's inspection process--will be discussed, with the viewpoint that the minimization process developed may be extended to those items returned by the consumer.

Producer Inspection

As illustrated by the decision-action diagram, the producer first faces the decision as to whether to accept the items as is or to subject them to some form of inspection. It will be assumed that the decision is made to inspect the items, and that a single sampling plan is adopted. It is also assumed that the items reach the final inspection stage in lots of a predetermined size N . Some percentage of the lots will be rejected. These lots will contain some number (variable) of defects a per lot. If defective items in the samples are replaced by good items, then a will be equal to the remaining number of defects in the lot; if not, then a will be equal to the total number of defects in the lot.

Given a rejected lot, the producer now has the three alternatives of scrapping the lot, screening it, or selling it at a reduced price. Although in real life it may be known that one or more of the alternatives do not exist or are so prohibitively expensive that they must be eliminated from consideration, it is assumed that all three alternatives are available.

Scrap Rejected Lots

It will be assumed that an excess demand situation exists for the producer; i.e., all lots that are not rejected may be sold. Thus if a rejected lot is scrapped, the resultant loss in revenue will be $NC - J$, where C is the selling price of a good item, and J is the scrap value of the lot ($0 \leq J < C$). It is likely that J will be fairly constant for each lot (varying only with time), and here it will be considered as constant for all rejected lots. The difference $NC - J$ will be regarded as the additional cost (over and above the costs of production and inspection) of having a rejected lot, although it is, strictly speaking, a loss in revenue. Thus the cost of adopting the alternative of scrapping a rejected lot is $NC - J$.

Sell Rejected Lots at Reduced Price

It may be known that rejected lots can be sold at some reduced price R per lot. If a lot is sold at the reduced price R , the resultant loss in revenue will be $NC - R$. Again, this loss in revenue will be regarded as a cost--the additional cost to the producer of having a lot rejected and electing to sell it at a reduced price.

The consumer purchases a reduced price lot from the producer with the knowledge that it has been rejected by the producer's inspection plan. The question now arises as to what the consumer's policy is toward rejected lots. It seems logical that he will not inspect them by the same procedure as that used for incoming good lots, but use them directly, perhaps screening them for defects beforehand. In this study, it will be assumed that no items from a reduced price lot will be returned to the producer. The consumer pays a reduced price for the lots, but has to accept them as is.

Screen Rejected Lots

The screening, or 100 per cent inspection of rejected lots, will involve additional cost above that of production and inspection, due to additional man-hours, facilities, etc. Let the estimated cost of screening an item be s . Then the cost of screening a lot is $(N-n)s$, where n is the number of items sampled in the final inspection. The defective items in each rejected lot are replaced by good ones, and the lot then in theory contains 100 per cent good items. It is assumed that any fraction defective in the lot due to faulty screening will be small enough so that a negligible number of items face the possibility of being returned by the consumer.

After sampling and screening a total of a defective items will be obtained. Since there is no method of determining the exact value of a for each lot beforehand, some procedure must be adopted for predicting or estimating an average value for a . Various procedures for determining $E(a)$ will be considered later.

The producer may now scrap the defective items, rework them, sell them at a reduced price, or utilize some combination of the three alternatives. Assuming that all three alternatives exist, a procedure for minimizing the cost of handling defective items obtained by screening will be developed.

Scrap Defective Items

Assume that each scrapped item has some salvage value j , where j is constant for all defective items, regardless of the type of defect ($0 \leq j < C$). The additional "cost" of scrapping defective items will be $a(C-j)$, and the total additional cost will be

$$(N-n)s + a(C-j) \quad (4-1)$$

Sell Defective Items at Reduced Price

Assume that each defective item may be sold for some reduced price c which depends on the type of defect (where each item has only one defect). The primary objective is to separate defective items into clearly identifiable categories, to which reduced prices may be assigned. The categories must be such that a defective item may be classified in one and only one category. In this study, categories will be considered to be defect types, and consequently each item may have only one type of defect. Units with different types of defects may have different reduced prices. Let \bar{c} be the average reduced price per item:

$$\bar{c} = \sum_{i=1}^k a_i c_i / a \quad (4-2)$$

where

k = total number of different defect types.

a_i = number of units with defect type i .

c_i = reduced price for unit with defect type i .

a = total number of defective items per lot.

The total additional cost will now be

$$(N-n)s + a(C-\bar{c}) \quad (4-3)$$

for a rejected lot, if the decision is made to screen the lot and sell defective items at reduced prices. It is assumed that no defective items purchased from the producer at reduced prices will be returned.

Methods for obtaining average values for the a_i 's will be discussed in Chapter V.

Rework Defective Items

Consider an average cost \bar{r} to rework a defective item:

$$\bar{r} = \sum_{i=1}^k a_i r_i / a \quad (4-4)$$

where

k = total number of different defect types.

a_i = number of units per lot with defect type i .

r_i = rework cost for unit with defect type i .

The total additional cost per lot will be

$$(N-n)s + a\bar{r} \quad (4-5)$$

Scrap, Rework, and Reduced Price Combination

It is possible to utilize a combination of the three alternatives. Consider an expression which will combine all three alternatives. Assume that:

Items with defect types $i = 1, 2, \dots, h$ are scrapped.

Items with defect types $i = h+1, h+2, \dots, m$ are reworked.

Items with defect types $i = m+1, m+2, \dots, k$ are sold at reduced prices.

Now

$$\text{Scrap cost/lot} = a_s(C-j)$$

$$\text{Rework cost/lot} = a_r \bar{r}'$$

where

$$\bar{r}' = \sum_{i=h+1}^m a_i r_i / a_r$$

$$\text{Reduced price cost/lot} = a_e(C-\bar{c}')$$

where

$$\bar{c}' = \sum_{i=m+1}^k a_i c_i / a_c$$

and

$$a_s = \sum_{i=1}^h a_i = \text{total number of defective items scrapped.}$$

$$a_r = \sum_{i=h+1}^m a_i = \text{total number of defective items reworked.}$$

$$a_c = \sum_{i=m+1}^k a_i = \text{total number of defective items sold at reduced price.}$$

$$a_s + a_r + a_c = a$$

The total additional cost for a rejected lot will be

$$(N-n)s + a_s(C-j) + a_r \bar{r}' + a_c(C-\bar{c}') \quad (4-6)$$

Now the *average additional cost* for rejected lots that are screened is

$$(N-n)s + \bar{a}_s(C-j) + \bar{a}_r \bar{r}' + \bar{a}_c(C-\bar{c}') \quad (4-7)$$

The above expression may be reduced to that for each of the three alternatives separately, or to any combination of two of them.

The Minimum-Cost Procedure

There are now a maximum of 3^k combinations of the three alternatives to be considered in the case where rejected lots are screened. One (or more) of these will yield a minimum value for (4-7). However, it is not necessary to consider all the combinations. Assuming that the policy selected for items of a particular type of defect is uniform for all items in that category, a minimum-cost procedure for the screening case may be determined by inspection. For each defect type i , select $\min (C-j, r_i, C-c_i)$. Since the alternative selected is uniform for all items in each grouping, the selection of the minimum cost figure for each type of defect will result in a minimum total additional cost for the screening case.

Now the costs of the three initial alternatives are compared.

These are:

1. NC - J (scrap rejected lots).
2. NC - R (sell rejected lots at reduced price).
3. $(N-n)s + \bar{a}_s(C-j) + \bar{a}_r\bar{r}' + \bar{a}_c(C-\bar{c}')$ (screen lots--minimum-cost procedure).

The minimum of these three values is chosen, and the procedure corresponding to this value is adopted.

The above procedure is very simple to apply, and is primarily an extension of the intuitive process utilized by many manufacturers. In a large number of situations, conditions may be such that the task of

obtaining cost estimates and price figures will not be justified by the savings incurred, or simple enough such that a management decision based upon available data will result in the same minimum cost. However, in situations where the volume of production is high and a fairly large number of defect types exist, the previously described procedure would probably be justified.

CHAPTER V

ESTIMATING THE AVERAGE NUMBER OF DEFECTS AND THE FREQUENCY OF EACH TYPE OF DEFECT IN REJECTED LOTS

Introduction

As mentioned in Chapter IV, it is necessary to devise some method of predicting or estimating the average number of defects and the frequency of each type of defect in rejected lots, in order that the alternative of screening and subsequent action be taken into consideration. This chapter will be concerned with means of obtaining these necessary values.

Estimating the Average Number of Defects in Rejected Lots for Single Sampling Plans

Given certain facts about the prior distribution of the number of defects X in lots and the nature of the single sampling plan, it is possible to obtain an average value for α , the number of defective items in a rejected lot, by analytical means. Two cases will be considered--sampling where α equals the total number of defectives in a rejected lot, and sampling where α equals the number of defectives remaining in the unsampled portion of the rejected lot.

Case I

α = total number of defectives in rejected lots.

This case corresponds to the situation where the sample, including defective items in the sample, is returned to the lot. In an accepted lot, it would be logical to replace the defective items obtained by sampling with good ones. However, since this discussion is concerned with rejected lots only, it will be assumed that defectives are not replaced with good items and the sample is returned to the lot unaltered.

Let X equal the number of defective items in a lot, and $f_N(X)$ be the prior distribution of X .

$$f_N(X) = \text{Prob}(\text{lot of size } N \text{ has } X \text{ defectives}) \quad (5-1)$$

where

$$X = 0, 1, 2, \dots, N$$

Now consider a single sampling plan having a known operating characteristic function described by (N, n, c) , where n is the sample size and c is the acceptance number. For a given sample of size n , the lot from which the sample was taken will be accepted if x , the number of defectives in the sample, is less than or equal to c , and rejected if x is greater than c . The probability that a lot containing X defectives will be accepted is given by

$$P_a(X) = \sum_{x=0}^c \frac{\binom{X}{x} \binom{N-X}{n-x}}{\binom{N}{n}} \quad (5-2)$$

This expression may be used to calculate the probabilities from which an OC-curve for the plan may be drawn. For lot sizes greater than ten times the sample size, the probabilities are given adequately by the binomial approximation:¹

$$P_a(p) = \sum_{x=0}^c \binom{n}{x} p^x (1-p)^{n-x} \quad (5-3)$$

where

$$p = X/N$$

If the samples are also large, the normal or Poisson distributions can be used. Generally, if the lot fraction defective $p = X/N$ is such that $pN > 5$, the probabilities can be figured from the normal distribution with mean \bar{p} and standard deviation $\sqrt{\bar{p}(1-\bar{p})/N}$. If $pN < 5$, the Poisson distribution gives better results, the probability of acceptance being given by²

$$P_a(p) = \sum_{x=0}^c (np)^x e^{-np} / x! \quad (5-4)$$

where.

¹Hald, *op. cit.*, p. 279.

²Acheson J. Duncan, *Quality Control and Industrial Statistics*, Homewood, Illinois, Richard D. Irwin, Inc., 1965, p. 133.

$$p = X/N$$

For the purposes of the discussion, the general (hypergeometric formulation) will be used.

Now the probability that a lot with X defectives will be rejected is

$$P_r(X) = 1 - P_a(X) = \sum_{x=c+1}^n \frac{\binom{X}{x} \binom{N-X}{n-x}}{\binom{N}{n}} \quad (5-5)$$

The expected number of defectives in a rejected lot (for Case I) may be calculated by the following expression:

$$\bar{a} = E(X_{rej}) = \bar{X}_{rej} = \frac{\sum_X X f_N(X) P_r(X)}{\sum_X f_N(X) P_r(X)} \quad (5-6)$$

Case II

a = number of defectives in remainder of rejected lot.

This case is analogous to the situation where defective items are replaced with good ones, or where the sample is not returned to the lot.

Hald defines several expressions which are useful for the purposes of this discussion. Let¹

¹Hald, *op. cit.*, p. 284.

$P(x|X)$ = Prob(sample of size n from lot of size N containing
 X defectives will contain x defectives

$$= \frac{\binom{X}{x} \binom{N-X}{n-x}}{\binom{N}{n}} \quad (5-7)$$

It may be seen that

$$P_a(X) = \sum_{x=0}^c P(x|X) \quad (5-8)$$

Then the simultaneous distribution of X defectives in the lot and x defectives in the sample is given by¹

$$P(X, x) = f_N(X)P(x|X) \quad (5-9)$$

from which the marginal distribution of x ; i.e., the overall, or *average* probability of x defectives in the sample, is derived:¹

$$g_n(x) = \sum_X P(X, x) = \sum_X f_N(X)P(x|X) \quad (5-10)$$

The proportion of rejected lots is¹

$$\sum_{x=c+1}^n g_n(x) \quad (5-11)$$

¹*Ibid.*, p. 286.

It may now be seen that the expected number of defects in a sample from a lot that is rejected is

$$E(x_{\text{rej}}) = \bar{x}_{\text{rej}} = \sum_{x=c+1}^n x g_n(x) \quad (5-12)$$

An expression for \bar{X}_{rej} has been derived for Case I. Now the number of defective items remaining in the lot is simply $(X - x)$, and (from statistical theory which states that $E(a \pm b) = E(a) \pm E(b)$ if a and b are random variables with expectations¹⁾) it may be seen that

$$\bar{a} = E(X_{\text{rej}} - x_{\text{rej}}) = E(X_{\text{rej}}) - E(x_{\text{rej}}) = \bar{X}_{\text{rej}} - \bar{x}_{\text{rej}} \quad (5-13)$$

Thus analytical expressions for determining \bar{a} for both cases have been developed.

Prior Distributions

There are a number of prior distributions discussed in the literature. The general nature of each prior distribution will be discussed here, and $g_n(x)$ will be specified in each case, in view of its relevance in calculating the average number of defects in rejected lots for Case II.

Two-Point Binomial

Consider the distribution $f_N(X)$ such that:

¹William Feller, *An Introduction to Probability Theory and Its Application*, New York, John Wiley & Sons, 1966, p. 208.

$$f_N(X) = \begin{cases} w_1 & \text{for } X = X_1 \\ w_2 & \text{for } X = X_2 \\ 0 & \text{for all other } X \end{cases} \quad X = 0, 1, 2, \dots, N \quad (5-14)$$

For this relatively simple distribution

$$g_n(x) = w_1 \frac{\binom{X_1}{x} \binom{N-X_1}{n-x}}{\binom{N}{n}} + w_2 \frac{\binom{X_2}{x} \binom{N-X_2}{n-x}}{\binom{N}{n}} \quad (5-15)$$

Hypergeometric

Consider the situation where a stock of the same size and composition always exists, namely M items containing $A = M\bar{p}$ defectives, and that lots of N items are selected at random from this stock. The prior distribution is then¹

$$f_N(X) = \frac{\binom{A}{X} \binom{M-A}{N-X}}{\binom{M}{N}} \quad (5-16)$$

$$= \frac{\binom{N}{X} \binom{M-N}{A-X}}{\binom{M}{A}} \quad (5-17)$$

The expression for $g_n(x)$ for the hypergeometric distribution and for several other important distributions may be determined from a theorem of Hald's.²

Let X denote the number of elements having a certain attribute in a population of N elements and let x and $y = (X-x)$ denote the corresponding numbers of elements in a random sample (drawn

¹Hald, *op. cit.*,

Ibid., p. 299.

without replacement) of size n and in the remainder of the population, respectively. If the distribution of X is a hypergeometric, a binomial, a rectangular, a Polya, or a mixed binomial distribution, or any weighted average of these distributions with weights independent of N and X , then for any N the distribution of x is the same as the distribution of X with n substituted for N , and the distribution of y for given x is also of the same type but with parameters depending on x and n .

Thus for the hypergeometric case

$$g_n(x) = \frac{\binom{A}{x} \binom{M-A}{n-x}}{\binom{M}{n}} \quad (5-18)$$

Binomial

Let the lots to be inspected be produced by a process in control with process average equal to \bar{p} so that¹

$$f_N(X) = \binom{N}{X} \bar{p}^X (1-\bar{p})^{N-X} \quad (5-19)$$

Hald¹ considers the compound probability $P(x, y)$, where $y = X - x$. He shows that

$$P(x, y) = f_n(x; \bar{p}) f_{N-n}(y; \bar{p}) \quad (5-20)$$

which means that x and y are stochastically independent and that both variables are binomially distributed with parameters specified by the theorem mentioned earlier. Thus

¹*Ibid.*, p. 297.

$$g_n(x) = \binom{n}{x} \bar{p}^x (1-\bar{p})^{n-x} \quad (5-21)$$

where

$$p = \bar{x}/n$$

Other distributions, such as the Polya and mixed binomial, are discussed in the literature. For all of the prior distributions encountered, the final expressions for the number of defects in rejected lots (Cases I and II) are lengthy and somewhat complicated, and do not simplify to any significant degree. For most prior distributions the process of calculating the average number of defects in rejected lots by hand is so tedious as to be impractical from the producer's viewpoint. There are two alternative methods of determining \bar{a} :

1. Utilize existing computer facilities and prepare a program for calculating \bar{a} .
2. Determine an estimate of \bar{a} by actually screening rejected lots, or by simulating, sampling and screening lots utilizing Monte Carlo techniques.

Estimating the Average Number of Defects in Rejected Lots by Sampling and Screening Simulated Lots

It was desired to test the closeness of results obtained by simulating and sampling lots to those predicted by the analytical procedures for Cases I and II.

A simple prior distribution was used, and Monte Carlo simulation was used to generate lots.¹ The lots were sampled randomly according to a single sampling plan with specified parameters, and the average number of defective items in rejected lots for both Cases I and II was tabulated.

Prior Distribution

The prior distribution was a two-point binomial, specifically:

$$f_N(X) = \begin{cases} w_1 & \text{for } X = X_1 \\ w_2 & \text{for } X = X_2 \\ 0 & \text{for all other } X \end{cases}$$

$$X = 0, 1, 2, \dots, 1000$$

$$X_1 = 20, \quad X_2 = 219$$

$$w_1 = 0.90, \quad w_2 = 0.10$$

Single Sampling Plan

The single sampling plan selected may be described by the following parameters:

$$N = 1000.$$

$$\text{Producer's risk point} = (p_1, \alpha) = (0.02, 0.05).$$

$$\text{Consumer's risk point} = (p_2, \beta) = (0.219, 0.10).$$

$$c = 1.$$

¹Ernesto Ruiz Pala, Carlos Avila-Beloso, and William W. Hines, *Waiting-Line Models, An Introduction to Their Theory and Application*, New York, Reinhold Publishing Corporation, 1967, pp. 107-138.

$n = 18.$

Procedure

1000 lots were generated by the Monte Carlo simulation procedure. A random sample was drawn from each lot, and the lot was rejected or accepted according to the criteria of the plan. A tabulation was made of the number of defective items in rejected lots.

Results

Figure 5-1 illustrates the cumulative tabulated average number of defective items per rejected lot as a function of the number of lots sampled, for Case I. The horizontal line represents the expected value calculated by the analytical procedure. Figure 5-2 presents the cumulative average as a function of the number of rejected lots, and Figure 5-3 shows the same relationship for Case II.

Analysis of Results

It may be seen that the tabulated value in each case fluctuates about the true value and approaches it as the maximum number of lots sampled (or rejected lots screened) is reached. The natural conclusion that may be drawn is that the cumulative effect will cause the tabulated figure to approximate the true value more closely as more lots are sampled. Since analytical confidence intervals were not derived for estimates of \bar{a} by sampling simulated (or actual) lots, any recommendations as to the number of lots to be sampled (or rejected lots to be screened) must be somewhat arbitrary in nature. Keeping this restriction in mind, this writer will suggest an arbitrary figure of 200 as the *minimum number of rejected lots to be screened* in order to insure 95 per cent confidence regarding the true value of \bar{a} .

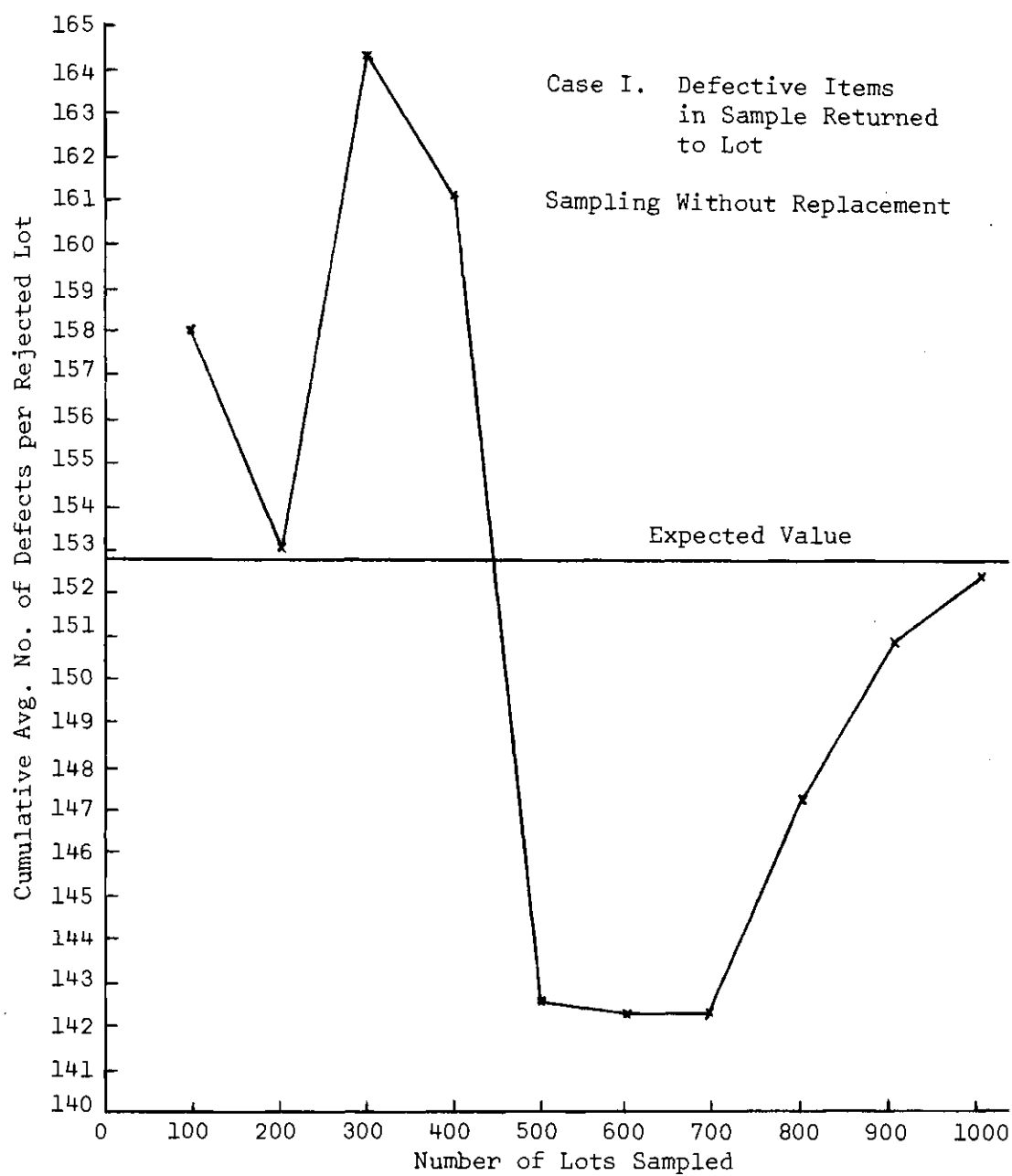


Figure 5-1. Average Number of Defective Items per Rejected Lot as a Function of the Number of Lots Sampled

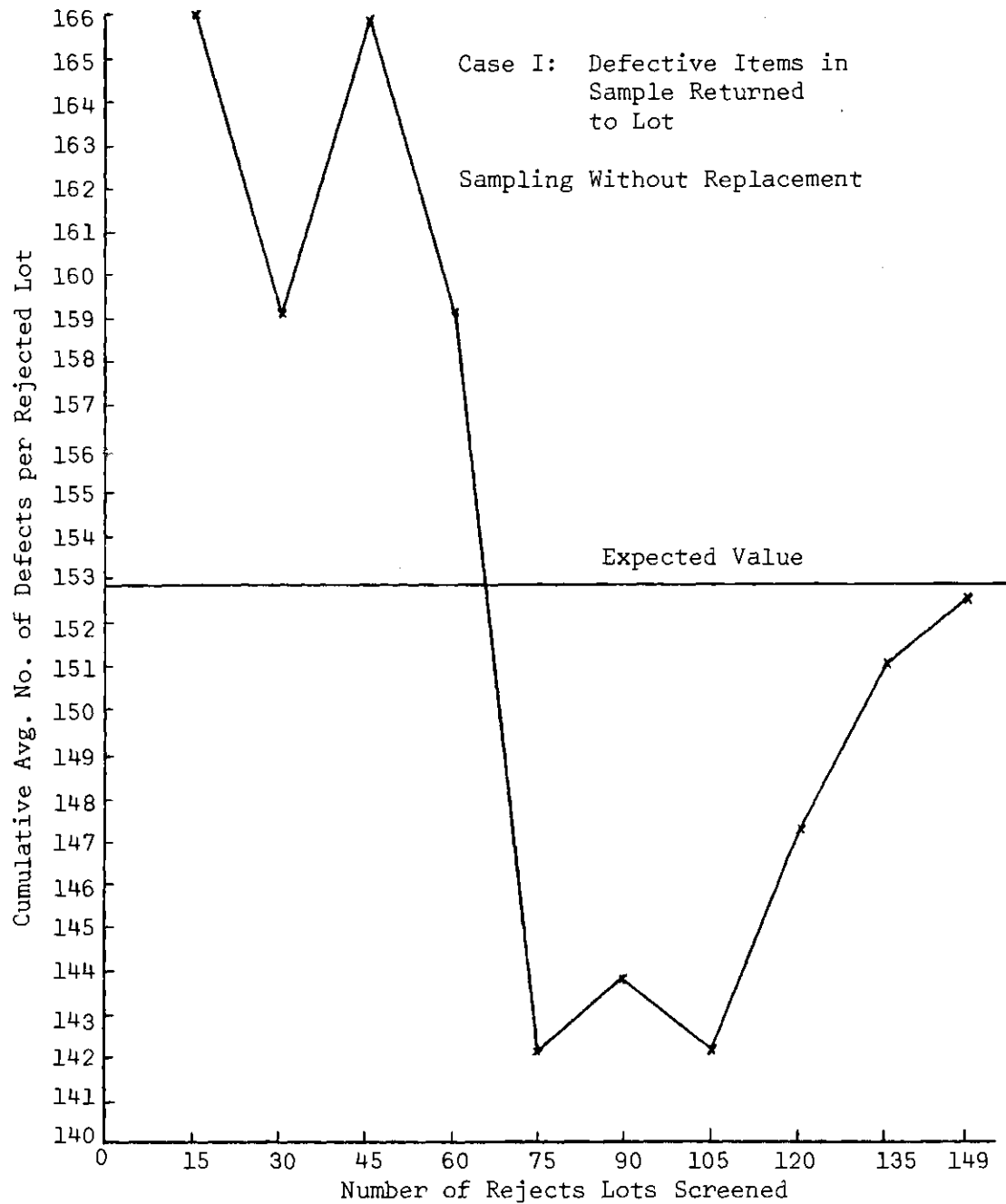


Figure 5-2. Average Number of Defective Items per Rejected Lot as a Function of the Number of Rejected Lots Screened

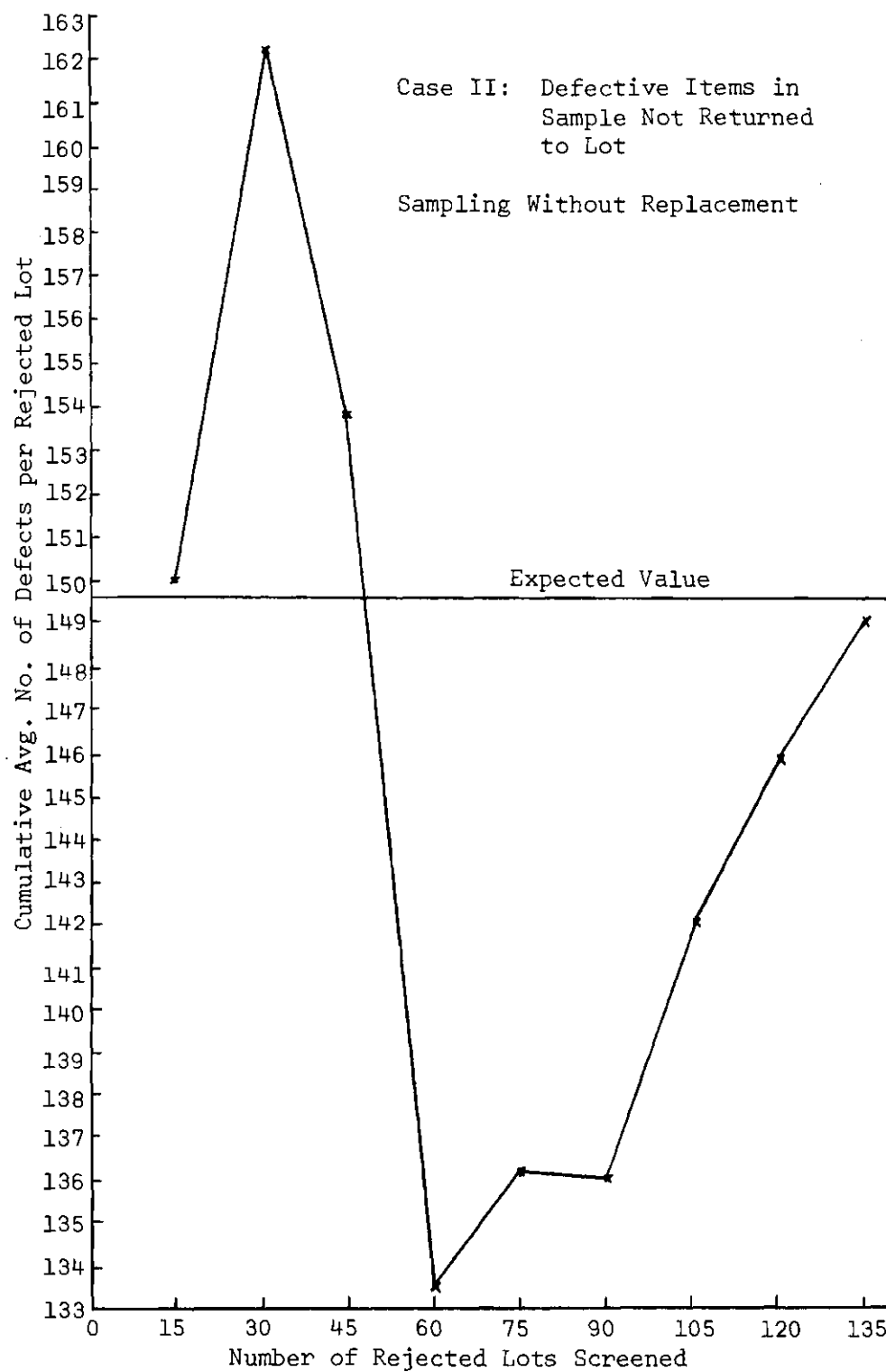


Figure 5-3. Average Number of Defective Items per Rejected Lot as a Function of the Number of Rejected Lots Screened

Estimating the Frequency of Each Type
of Defect in Rejected Lots

In Chapter IV the average minimum-cost expression for the alternative of screening and subsequent action was given by

$$\text{Cost} = (N-n)s + \bar{a}_r \bar{r}' + \bar{a}_c (C-\bar{c}) + \bar{a}_g (C-j)$$

It may be seen that this cost is dependent upon the values of the a_i 's--the frequencies of the different types of defects. Therefore, it is necessary to devise some method of predicting the average values of the a_i 's.

In the previous section, \bar{a} was estimated by arbitrarily screening a sample of rejected lots. It was suggested that a minimum number of 200 rejected lots be screened in order to insure 95 per cent confidence. Since no analytical procedure for obtaining confidence intervals for \bar{a} was developed, the simple rule of sampling more rejected lots to obtain a more accurate estimate was recommended. It is not likely that the prior distribution of each type of defect will be known, and therefore the producer must resort to estimating the average values of the a_i 's from sample data. It will be necessary to screen and sort actual lots to obtain these average values. Again, the procedure of sampling more rejected lots to obtain greater accuracy is recommended. However, since each a_i is less than a , for a given number of screened lots the accuracy of the estimate of each \bar{a}_i will be less than the estimate of \bar{a} . If the producer desires very accurate estimates of the \bar{a}_i 's, he should plan to screen a sample of rejected lots considerably larger than the arbitrary

number of 200 mentioned earlier; if he is willing to tolerate less accurate estimates and settle for a reasonably close approximation of \bar{a} , then the figure of 200 might be regarded as sufficient.

Methods of Tabulating Data

Some method of tabulating data may easily be devised by the producer. For example, the types of data collection sheets illustrated by Figures 5-4 and 5-5 might be used.

Adjusting the \bar{a}_i 's for a Known Analytical Value of \bar{a}

If it is possible for the producer to determine the value of \bar{a} analytically, then it is suggested that each \bar{a}_i obtained by sampling be multiplied by the factor $\bar{a}_i/\bar{a}_{\text{est}}$, so that the new \bar{a}_i 's will retain their proportionality but sum to \bar{a} rather than \bar{a}_{est} .

A Revised Cost Expression

The assumption is made that for each \bar{a}_i , the ratio \bar{a}_i/\bar{a} is constant over all values of \bar{a} ; i.e., in the long run each \bar{a}_i is some fixed percentage of \bar{a} , regardless of the value of \bar{a} . With this assumption in mind, it is possible to define a *minimum average cost per item* of handling defective items after screening, *independent* of the value of \bar{a} :

$$\bar{c}_{\min} = (\bar{a}_p r' + \bar{a}_c (C - \bar{c}') + \bar{a}_g (C - j)) / \bar{a} \quad (5-22)$$

The average minimum cost \bar{c}_{\min} will be constant over all values of \bar{a} . Now the average minimum total cost of screening may be rewritten in the following form:

$$\text{Screening cost} = (N - n)s + \bar{a}\bar{c}_{\min} \quad (5.23)$$

Rejected Lots		
Screening Process		
Defective Types and Frequencies		
Lot Tally Sheet		
Production Data on Lot _____		
Defect Type	Frequency	Number
1		
2		
3		
.		
.		
k		
		Total

Figure 5-4. Proposed Data Tabulation Sheet--Lot Tally Sheet

<p>SUMMARY SHEET</p> <p>Rejected Lots</p> <p>Screening Process</p> <p>Defective Types and Frequencies</p> <p>Production Data on Lots _____</p> <p>No. of Lots Screened _____</p> <p>Price/Unit _____</p> <p>Scrap Value/Unit _____</p>				
Defect Type	Number	Avg./Lot	Rew. Cost/Unit	Red. Pr./Unit
1				
2				
3				
.				
.				
k				
Total				

Figure 5-5. Proposed Data Tabulation Sheet--Summary Sheet

It is seen that the magnitude of the screening cost, assuming fixed sample size, varies directly with a (the reason for assuming fixed sample size will be discussed in Chapter VI). The cost expression shown above will be used in place of the original expression in the next chapter.

CHAPTER VI

MINIMUM-COST MODELS INCORPORATING CONSUMER BEHAVIOR

Consumer Behavior

Some policy may exist by which the consumer returns defective lots or items to the producer. These consumer-rejected lots or defective items (partial lots) are combined with the lots rejected by the producer himself to form his total stock of defective material.

The minimum-cost procedure discussed previously assumes that complete lots exist initially. According to the consumer's policy toward returning defective material, the producer's stock of defective material may consist of complete lots only, or of complete lots and partial lots. Partial lots, or groups of defective items, may not be subjected to the entire minimum-cost decision process, since the process presupposes complete lots, and must enter the process at a later point (as shown by the decision-action diagram). If only complete lots are returned by the consumer, the entire decision process may be applied to all defective material (since all defective material will be in lots). The case where only complete lots are returned by the consumer is given primary attention in the discussion.

According to the decision-action diagram, there are five different cases in which material rejected by the consumer may be returned to the producer:

Case 1--Incoming lots are inspected by the consumer, and rejected

lots are returned to the producer. No defective items discovered during use of accepted lots will be returned.

Case 2--Same as Case 1, except that defective items from accepted lots are returned.

Case 3--Incoming lots are inspected and rejected lots are screened, the defective items being sent back to the producer. No defective items from accepted lots are returned.

Case 4--Same as Case 3, except that defective items from accepted lots are returned.

Case 5--Incoming lots are accepted without inspection, and defective items discovered during use are returned.

Each case will now be considered in more detail.

Case 1

The consumer submits incoming lots to a single sampling plan with known (n', c') , where n' is the sample size and c' is the acceptance number for the plan. All rejected lots are returned (sample included); all accepted lots are used.

The producer now has lots that have been rejected by his sampling plan, and lots that have been accepted by him but rejected and returned by the consumer. Since all of his unacceptable material is in lots, he may subject the entire stock to the complete decision process.

The number of unacceptable lots which the producer has on hand at a given time will be the sum of the number rejected by his own sampling plan and the number rejected and returned by the consumer.

It is of economical interest to the producer to minimize the total number or rejected lots on hand. This may be done by minimizing the probability of obtaining a rejected lot. An expression for the probability of obtaining a rejected lot will be developed, and procedures for minimizing this probability will be discussed.

As discussed previously, the minimum-cost procedure for handling rejected lots involves basically a choice between three alternatives--scrap rejected lots, sell them at a reduced price, or screen them and take minimum cost action with defective items. If rejected lots are scrapped or sold at a reduced price, the resultant loss in revenue is assumed to be independent of the number of defects in the lots; i.e., a single scrap value or reduced price exists for the lots. However, the minimum cost of screening and subsequent action is dependent upon the number of defective items in the rejected lots. An analytical procedure has been specified for determining the average number of defects in a lot rejected by the producer's inspection plan. A procedure which includes both the producer's and consumer's inspection plans will be developed.

Probability P(R) of Obtaining a Rejected Lot

Consider the modified prior distribution of lots received by the consumer. If it is assumed that the producer does *not* replace defective items in the sample with good ones before sending the accepted lot to the consumer, then the modified prior distribution will be

$$f_N'(X) = f_N(X)P_a(X) \quad (6-1)$$

However, it is more reasonable to assume that defective items in the sample are replaced by good ones. Given an accepted lot with X defectives, the expected number of defective items $E(x_a)$ in the sample will be

$$E(x_a) = \bar{x}_a = \sum_{x=0}^c x g_n(x) \quad (6-2)$$

and thus the modified prior distribution $f_N'(X')$ is given by

$$f_N'(X') = f_N(X) P_a(X) \quad (6-3)$$

where

$$X' = X - \bar{x}_a \quad (6-4)$$

It is seen that X' is not likely to be an integer, and thus the hypergeometric distribution may not be used to calculate the probabilities of acceptance for the consumer. It is necessary to approximate the probabilities of acceptance by means of the binomial distribution.

Let $p' = X'/N$. Now $f(p') = f_N'(X')$, and the probability that the consumer will accept (using a single sampling plan) an incoming lot with fraction defective p' is given by

$$P_a(p') = \sum_{x'=0}^{c'} \binom{n'}{x'} (p')^{x'} (1-p')^{n'-x'} \quad (6-5)$$

where

n' = size of sample for consumer's plan.

c' = acceptance number for consumer's plan.

Likewise, the probability that the consumer will reject an incoming lot with fraction defective p' is given by

$$P_r(p') = 1 - P_a(p') = \sum_{x'=c'+1}^{n'} \binom{n'}{x'} (p')^{x'} (1-p')^{n'-x'} \quad (6-6)$$

Now the probability that the producer will obtain a rejected lot may be determined as follows:

$$\begin{aligned} P(R) = \text{Prob}(\text{rejected lot}) &= \text{Prob}(\text{lot rejected by producer}) \\ &+ \text{Prob}(\text{lot accepted by producer}) \times \text{Prob}(\text{lot accepted} \\ &\text{by producer is rejected by consumer}) \end{aligned}$$

As stated before, the proportion of lots rejected by the producer is

$$\sum_{x=c+1}^n g_n(x)$$

where

$$g_n(x) = \sum_X P(x/X) = \sum_X \binom{N}{X} \binom{N-X}{n-x} / \binom{N}{n}$$

Likewise, the proportion of lots accepted by the producer is

$$\sum_{x=0}^c g_n(x)$$

The proportion of incoming lots rejected by the consumer is

$$\sum_{x'=c'+1}^{n'} g_n(x') \quad (6-7)$$

where

$$g_n(x') = \sum_{X'} P(x' | X')$$

and

$$P(x' | X') = P(x' | p') = \binom{n'}{x'} (p')^{x'} (1-p')^{n'-x'}$$

The probability of a rejected lot may thus be expressed mathematically as:

$$P(R) = \sum_{x=c+1}^n g_n(x) + \left[\sum_{x=0}^c g_n(x) \right] \left[\sum_{x'=c'+1}^{n'} g_n(x') \right] \quad (6-8)$$

Expected (Average) Number of Defects in an Unacceptable Lot

The average number of defective items in an unacceptable lot which the producer has on hand will be a weighted average of the value for lots rejected by himself and the value for returned lots. Redefining \bar{a} as the average number of defective items in any unacceptable lot:

\bar{a} = (proportion of lots rejected by producer) \times (average number of defects in lots rejected by producer) + (proportion of lots accepted by producer) \times (proportion of lots accepted by producer that are rejected by consumer) \times (average number of defects in lots rejected by consumer).

It will be recalled that the average number of defective items in a producer-rejected lot (assuming Case I) was found to be

$$\bar{x}_{\text{rej}} = \frac{\sum_X x f_N(x) P_r(x)}{\sum_X f_N(x) P_r(x)}$$

Likewise, for the consumer's situation

$$\bar{x}'_{\text{rej}} = \frac{\sum_{X'} x' f_{N'}(x') P_r(x')}{\sum_{X'} f_{N'}(x') P_r(x')} \quad (6-9)$$

$$= \frac{\sum_{p'} (Np') f(p') P_r(p')}{\sum_{p'} f(p') P_r(p')}$$

$$= \frac{\sum_{p'} p' f(p') P_r(p')}{\sum_{p'} f(p') P_r(p')}$$

Thus

$$\bar{a} = \left[\sum_{x=c+1}^n g_n(x) \right] \left[\bar{x}_{rej} \right] + \left[\sum_{x=0}^c g_n(x) \right] \left[\sum_{x'=c'+1}^{n'} g_n(x') \right] \left[\bar{x}'_{rej} \right] \quad (6-10)$$

Minimizing the Total Failure Cost for Case 1

In the previous section, expressions were derived for determining the probability of a rejected lot and the average number of defective items per lot. These expressions will be utilized in the process of minimizing total failure cost for Case 1.

Consider the failure cost per lot per lot *produced* (given that all lots are inspected by the producer). This cost will be

$$\text{Failure cost/lot} = P(R) \times (\text{Failure cost per rejected lot})$$

As stated before, there are three basic alternative for handling rejected lots--scrap, sell at a reduced price, or screen and take minimum-cost action with defective items. The three alternative failure costs per lot produced will thus be:

$$(1) \quad P(R) \times (NC - J)$$

$$(2) \quad P(R) \times (NC - R)$$

$$(3) \quad P(R) \times ((N-n)s + \bar{a}\bar{c}_{\min})$$

It was assumed previously that $(NC - J)$ and $NC - R$ were constants, independent of the number of defects. Therefore, (1) and (2) may be minimized by determining the minimum value of $P(R)$. However, it is only necessary to consider one of the two, namely $\min((NC - J), (NC - R))$, and then calculate

$$P(R)_{\min} \times \min((NC - J), (NC - R)) \quad (6-11)$$

$$= P(R)_{\min} \times \text{constant} \quad (6-12)$$

Now (3) is dependent upon *two* values-- $P(R)$ and \bar{a} --which are not independent of each other. Minimizing $P(R)$ may not minimize the total expression, nor may minimizing \bar{a} . Thus it is necessary to minimize the total expression; i.e., calculate

$$\min[P(R) \times ((N-n)s + \bar{a}\bar{c}_{\min})] \quad (6-13)$$

The OC-Curve and Parameters of the Single Sampling Plan

The primary means of describing a specific sampling plan is its OC-curve. The OC-curve is the plot of probability of acceptance versus lot fraction defective. Figure 6-1 illustrates a typical OC-curve for a single sampling plan.¹

¹Acheson, J. Duncan, *Quality Control and Industrial Statistics*, Homewood, Illinois, Richard D. Irwin, Inc., 1965, p. 134.

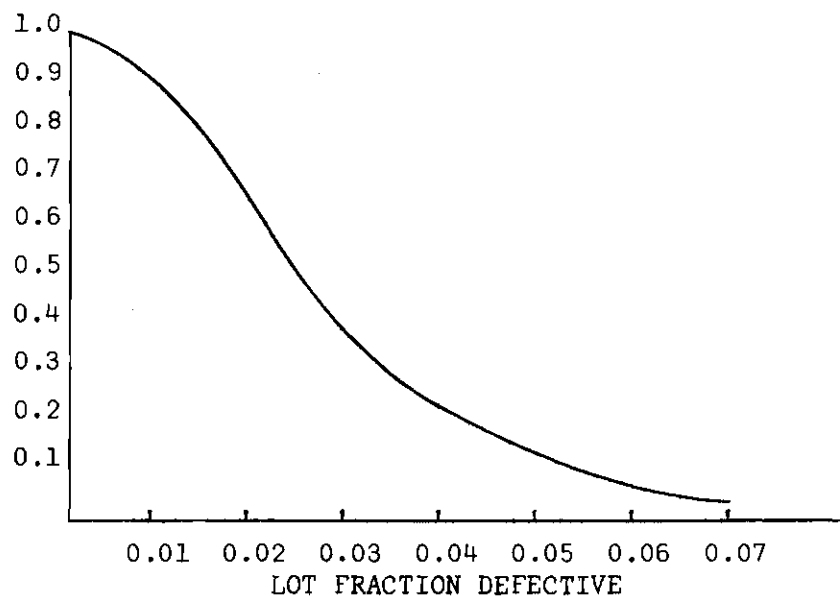


Figure 6-1. OC-Curve for the Single Sampling Inspection Plan with $n = 100$ and $c = 2$

The coordinates of each point on the OC-curve are $(p, P_a(p))$, where $p = X/N$ and

$$P_a(p) = \sum_{x=0}^c \frac{\binom{X}{x} \binom{N-X}{n-x}}{\binom{N}{n}}$$

The ideal OC-curve that discriminated perfectly between good and bad lots would have a π -shape. This curve would run horizontally at a probability of acceptance of 1.0 until some fraction defective $p^* = X^*/N$ considered unacceptable was reached, at which point it would drop vertically to zero and remain there for all values of p greater than p^* . Thus perfect control over the quality of inspected material would be insured. However, such an ideal plan can never be attained except by 100 per cent inspection.

The shape of the OC-curve depends upon the parameters n and c , for fixed lot size N . Increasing the sample size (while increasing c proportionately) increases the precision of the plan; i.e., the closeness of its shape to the z -shape, as shown by Figure 6-2.¹ Varying c (for fixed n) changes the probability that a lot with a given fraction defective will be accepted. Increasing c "tightens" the plan, and the OC-curve is lowered. Increasing c causes the plan to become more lax, and the OC-curve is raised (Figure 6-3).²

It may be seen that $P(R)$ and \bar{a} are also dependent upon n and c , and varying these parameters will cause $P(R)$ and \bar{a} to change. Assuming a consumer single sampling plan with known, fixed (n', c') , the producer may vary n (keeping c proportional), c , or n and c to obtain minimum values for the two alternatives mentioned previously.

Increasing the sample size may result in an increase in the cost of carrying out the inspection; i.e., appraisal costs may rise. It would thus be necessary to consider the combined costs of appraisal and failure in the analysis. Since this study is concerned exclusively with failure costs, emphasis will instead be focused on varying c for fixed n (and thus fixed appraisal cost) to obtain a minimum failure cost.

The alternative of no inspection at all by the producer must be explained. In this case, $P(R)$ and \bar{a} become equivalent to the expressions developed earlier for the case where lots were submitted to only

¹*Ibid.*, p. 135.

²*Ibid.*, p. 136.

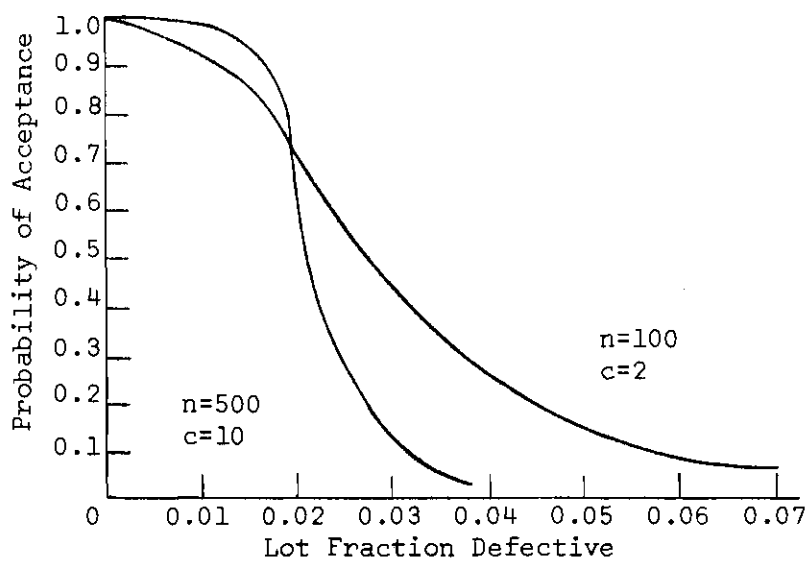


Figure 6-2. OC-Curves for Different Sample Sizes

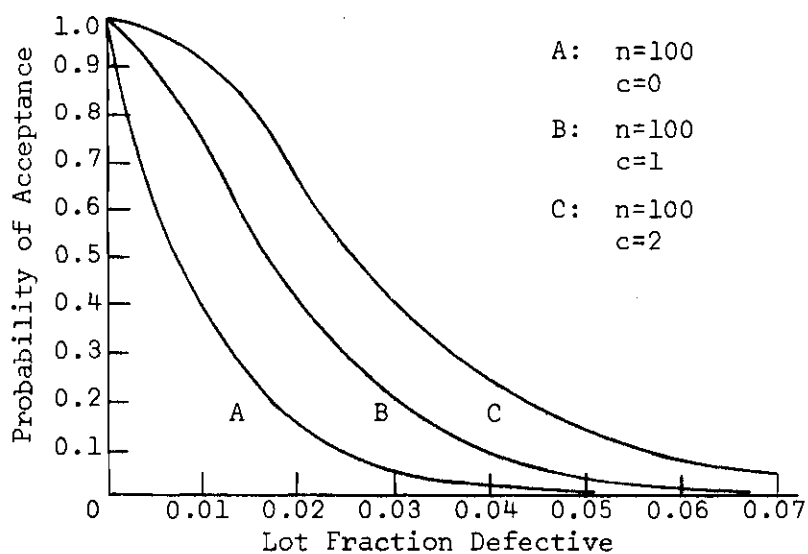


Figure 6-3. OC-Curves for Different Acceptance Numbers

one inspection plan (the producer's), substituting n' and c' for n and c . That is, for the case of no inspection by the producer

$$P(R) = \sum_{x=c'+1}^{n'} g_n(x) \quad (6-14)$$

and

$$\bar{a} = \frac{\sum_X X f_N(X) P_r(X)}{\sum_X f_N(X) P_r(X)} \quad (6-15)$$

The Minimum-Cost Procedure

The course of action which will minimize the total failure cost may now be determined graphically. Given knowledge of the (n', c') of the consumer's sampling plan, the producer may consider the following alternatives:

1. Do not inspect.
2. Set $c = 0, 1, 2, \dots, n$.

The minimum failure cost is obtained as follows:

(a) Plot $P(R)$ as a function of NI (no inspection) and $c = 0, 1, 2, \dots, n$, and select $P(R)_{\min}$.

(b) Select $\min((NC - J), (NC - R))$ and calculate

$$P(R)_{\min} \times \min((NC - J), (NC - R))$$

(c) Plot $P(R) \times ((N-n)s + \bar{a}\bar{c}_{\min})$ as a function of NI ,
 $c = 0, 1, 2, \dots, n$, and select

$$\min[P(R) \times ((N-n)s + \bar{a}\bar{c}_{\min})]$$

(d) Compare (b) and (c) and select the minimum value.

Thus a procedure has been determined for Case 1 which will select the course of action which minimizes total failure cost. Case 1 has been given particular attention because it is the belief of this writer that of the situations considered it is the one most likely to exist between the producer and consumer.

Case 2

Case 2 is concerned with the situation where lots rejected by the consumer are returned, and defective items from accepted lots are also returned.

It appears that Case 2 is not a logical course of action for the consumer to adopt. If the policy is to return all defective items, the inspection procedure will merely result in returning a large number of good items also. Since no defects are tolerated, the inspection process is of no value. Thus Case 2 will be disregarded from further consideration.

Case 3

The consumer inspects incoming lots, and all rejected lots are screened, rejected items being returned to the producer.

The producer may not subject returned items (partial lots) to

the entire decision process discussed previously, but must consider them at a later point in the process. The total resultant cost will be the sum of the costs of handling rejected lots and handling returned items.

As shown by the decision-action diagram, the producer may scrap returned defects, sell them at a reduced price, or rework them. The decision process is identical to that taken with defective items obtained by screening. If sorting of the returned items into the various classifications takes place, then a cost is incurred. This cost, on a per-item basis, will be assumed to be identical to the cost per item of screening and sorting rejected lots.

Assume that the same minimum-cost procedure is adopted for returned items as is utilized for defective items obtained by screening. Then the cost (per lot) of handling returned items is given by

$$a's + a_r' \bar{r}'' + a_p'(C - \bar{\sigma}'') + a_s'(C - j) \quad (6-16)$$

where a' , a_r' , a_p' , a_s' , r'' , and c'' correspond to a , a_r , a_p , a_s , r' , and c' , respectively, but pertain to returned items rather than defective items obtained by screening, and are on a "per lot size" rather than a "per item" basis.

It will be recalled that the average minimum cost $\bar{\sigma}_{\min}$ of handling a rejected item after screening was developed, and that $\bar{\sigma}_{\min}$ is constant for all values of $\bar{\alpha}$. Since returned defective items came originally from the same production process that yielded the defective items obtained by screening, and the same decision process is used for

both cases, it is logical to assume that \bar{c}_{\min} applies also to returned defective items, and that it is constant for all values of \bar{a}' . The average minimum cost (per lot size) of handling returned items is now given by

$$\bar{a}'s + \bar{a}'\bar{c}_{\min} = \bar{a}'(s + \bar{c}_{\min}) \quad (6-17)$$

where \bar{a}' is given by (6-9).

The producer may now use the same techniques mentioned in Case 1 to determine the following two expressions:

$$(a) \min\{P(R) \times [\min((NC - J), (NC - R)) + \bar{a}'(s + \bar{c}_{\min})]\}$$

$$(b) \min[P(R) \times ((N-n)s + \bar{a}\bar{c}_{\min} + \bar{a}'s + \bar{a}'\bar{c}_{\min})]$$

$$= \min[P(R) \times ((N-n+\bar{a}')s + (\bar{a}+\bar{a}')\bar{c}_{\min})]$$

The producer then selects the lesser value of (a) and (b), and takes the corresponding course of action to minimize his total failure cost.

Case 4

Case 4 is analogous to Case 2, except that only rejected items are returned, instead of rejected lots. Again, it appears that Case 4 is not a logical course of action for the consumer. The inspection process will be of no value to the consumer. Thus minimum-cost techniques for Case 4 will not be developed.

Case 5

The producer accepts incoming lots without inspection and returns all defective items.

Since all defective items are returned by the consumer, it is of economic value to the producer to detect lots with a high fraction defective, unless good will is a significant factor. The effects of good will (or lack of it) depend largely upon the particular producer-consumer relationship, and are difficult to evaluate on a monetary basis. Therefore, Case 5 will not be subject to analysis in this study.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

Concluding Remarks

Procedures have been developed for minimizing the failure cost to the producer in two different situations felt by this writer to be fairly common in real life. The research is oriented primarily toward high-volume manufacturing enterprises where it is known that the buyer will subject incoming lots of finished material to some form of statistical sampling inspection. Minimizing the failure cost may not result in the lowest possible over-all inspection cost, but the costs for the existing inspection process will be minimized and the effort involved in setting into motion a completely new inspection procedure will be eliminated.

Recommendations

There are a number of areas of possible interest extending the research in this study, which this writer lacked the time to pursue. The more important of these will be set down as recommendations for future research.

Computer Program

The development of a computer program to carry out the graphical techniques described in Chapter VI would facilitate the selection of the minimum failure cost inspection plan. The calculation process becomes fairly complex for prior distributions other than the simple

two-point binomial, and a general computer program adaptable to all prior distributions would be helpful.

Extension of the Models to Changes in the Sample Size

By considering the effects of changing the sample size n in addition to the acceptance number c for the producer's inspection plan, a plan may be selected which minimizes the sum of appraisal costs and failure costs, rather than failure costs alone. New cost expressions would be necessitated, which would incorporate the cost of inspecting an item. The model for minimizing total inspection costs would then fall in line with the general approach utilized by most authors in the literature.

Extension of the Models to Different Types of Inspection Plans

In the study, it was assumed that single sampling plans were used by both producer and consumer. It would be of interest to develop new cost models based on different combinations of statistical sampling plans (e.g., multiple sampling, sequential sampling) for the producer and consumer. An ultimate objective would be to derive a procedure for specifying the optimum *type* of plan for the producer to use and the optimum *parameters* for that plan, given the knowledge that the consumer is using a certain type of plan with known parameters.

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